NPS71-83-001

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A RANDOM TOUR PROCESS OF KNOWN LENGTH

BETWEEN KNOWN END POINTS

by

R. N. FORREST

January 1983

Approved for public release; distribution unlimited.

Prepared for: Strategic Systems Project Office, Arlington, VA 20376

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral J. J. Ekelund   D. A. Schrady
Superintendent   Provost

This report was prepared by:

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS71-83-001	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Random Tour Process of Known Length between Known End Points		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. N. Forrest		8. CONTRACT OR GRANT NUMBER(s) N0003083AF31319
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Strategic Systems Project Office 1931 Jefferson Davis Highway Arlington, VA 20376		12. REPORT DATE January 1983
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Random Track
Random Tour
Stochastic Process

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes a stochastic process that can be con-
sidered to describe a random track of predetermined length
that originates and terminates at predetermined end points.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE
1 JAN 73
S/N 0102-LF-014-6601

TABLE OF CONTENTS

TABLE OF CONTENTS

I.   Introduction

   This report describes a two dimensional stochastic process
and a program for its simulation. The process can be considered
to describe a random track of predetermined length that origi-
nates and terminates at predetermined end points. The process
is a modification of a process that is without the length and
termination constraints. This process, which is also described,
is a generalization of the random tour process described by
Washburn in Reference 1.

   A particular process is defined by the track length, the
maximum expected track segment length and the distance between
the end points. For motion at a constant speed, the track
length constraint is equivalent to a transit time constraint.

   In Section II, the process and the unmodified process are
described and some of their characteristics are discussed. In
Section III, the simulation program is described. By using
the program, graphs and data can be produced that represent
realizations of the process. The program, which is written in
BASIC, is for an HP-85.

II. A Description of the Process

The process, as it is described here, generates a random track of predetermined length that connects two predetermined end points. The track originates at a predetermined starting point, the first point, and terminates at a predetermined stopping point, the final point. The track is continuous between these two points and consists of a sequence of straight line track segments. Some representative tracks are shown in Figure 1.

A sequence of rectangular coordinate systems is used to define the process. The first is a reference system for the first track segment, the second is a reference system for the second track segment and so on. The x-axis of the first system is coincident with the line joining the first and final points, the origin is at the mid-point of the line and the system is oriented so that the final point is on its positive x-axis. For each successive system, the x-axis is coincident with the line joining the last point of the preceding track segment and the final point, the origin is at the mid-point of the line and the system is oriented so that the final point is on its positive x-axis. Each track segment is determined by two quantities: an angle θ and a length r. The angle θ is measured clockwise from the direction of the positive y-axis of the reference coordinate system of the track segment. It is the relative course that will generate the track segment. The length r determines the track segment length.
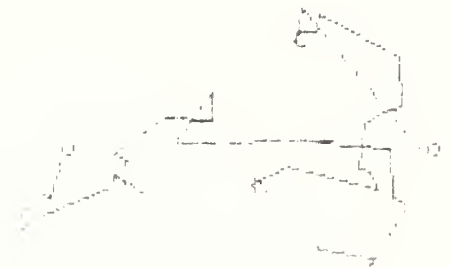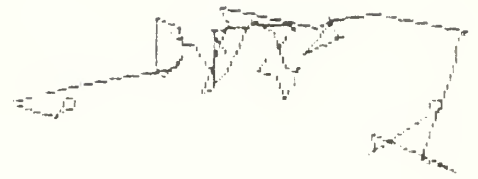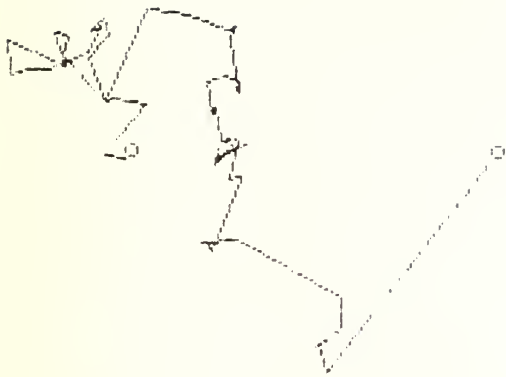
Figure 1. Some representative tracks. The tracks were generated
          by using the program that is described in Section III.

The process is defined as follows: For each track segment except the last a relative course θ is chosen from a distribution with density function

$$f_\Theta(\theta) = (1/2\pi)\, b/(a - c \sin \theta)$$

where $0 \le \theta < 2\pi$, $2c$ is the distance between the first and final points, $2a$ is the track length with $a > c \ge 0$ and $b = (a^2 - c^2)^{\frac{1}{2}}$. Next, a length r is chosen from a conditional exponential distribution with density function

$$f_{R|\Theta}(r|\theta) = [1/\delta(\theta)]\exp[- r/\delta(\theta)]$$

where $0 < r$, $\delta(\theta) = (a - c)\delta/(a - c \sin \theta)$ and δ is the maximum expected track segment length. Using the parameters a_k, b_k and c_k defined below, if $r \ge 2a_k$, then $r_k = b_k^2/(a_k-c_k \sin \theta_k)$. In this case, the $(k+1)$st track segment is the line that joins the last point of the kth track segment to the final point and the process terminates. If $r < 2a_k$, then $r_k = r$. In this case, the last point of the kth track segment and the final point determine the $(k+1)$st reference coordinate system and the process continues. The parameters in the above expressions are defined as follows: $2a_k$ is the remaining track length with $2a_1 = 2a$ and $2a_k = 2a_{k-1}-r_{k-1}$ for $k > 1$; $2c_k$ is the distance from the last point of the $(k-1)$st track segment to the final point with $2c_1 = 2c$ and $b_k = (a_k^2-c_k^2)^{\frac{1}{2}}$. The program that is described in Section III can be used to simulate this process.

Figure 2 illustrates the generation of a three segment track. The ellipse on which the second segment terminates
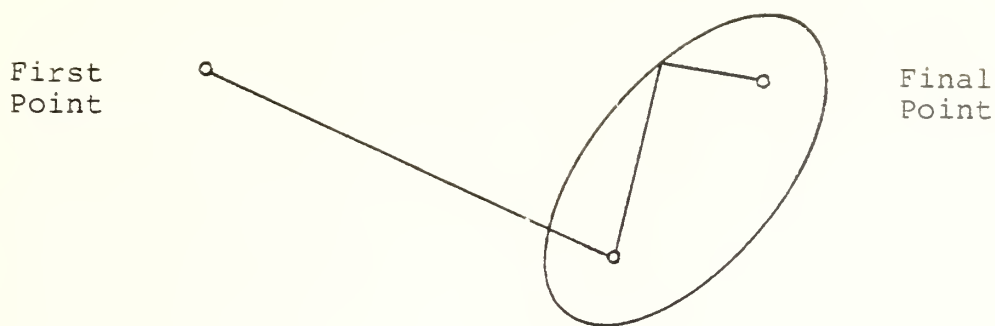
Figure 2. A three segment track and a focusing ellipse.
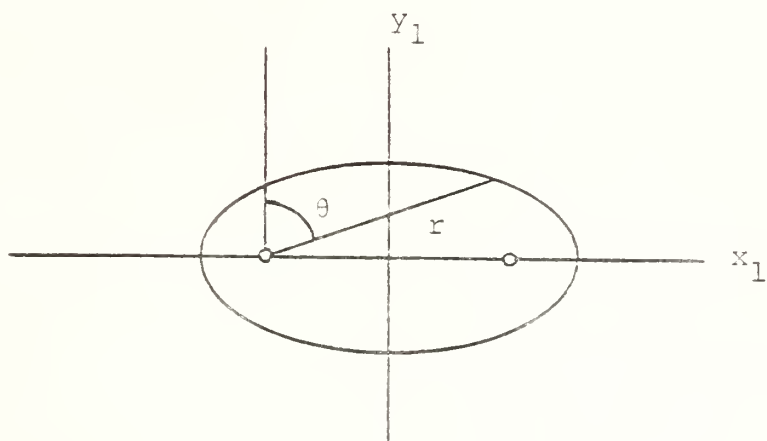

Figure 3. The first rectangular coordinate system, the
          associated polar coordinate system and the
          reference ellipse.

and the third segment begins has one focus at the last point of the first segment and the other at the final point. With $2c_2$ the distance between these two points, $2a_2 = 2a - r_1$ the remaining track length and $b_2^2 = a_2^2 - c_2^2$, the ellipse is defined through $x^2/a_2^2 + y^2/b_2^2 = 1$ in the second reference system. It is also defined by $r = b_2^2/(a_2 - c_2 \sin \theta)$ in the polar coordinate system with its origin at the last point of the first segment and its polar axis parallel to the y-axis of the second reference system.

As a basis for discussing the relative course and track length distributions that are associated with the process, consider the polar coordinate system with its origin at the first point and its polar axis parallel to the y-axis of the first reference system. Next, consider a reference ellipse with one focus at the first point and the other at the final point that is defined by $r = b^2/(a - c \sin \theta)$. The coordinate system and the ellipse are illustrated in Figure 3.

The distribution of the relative course θ and the conditional exponential distribution of the length R are both determined by $\rho(\theta) = (a-c)/(a - c \sin \theta)$, the ratio of the value of the radial coordinate of the ellipse at θ to its maximum value at $\pi/2$. Note that when a track segment's relative course is $\pi/2$ the segment is directed toward the final point. The distribution of the relative course θ is determined by the requirement that the ratio of the probability that a relative course is between θ and $\theta + \Delta\theta$ to $\Delta\theta$ is proportional to $\rho(\theta)$. The

conditional exponential distribution of the length R is determined by the requirement that the ratio of $\delta(\theta)$, the expected track segment length for θ, to δ, the expected track segment length for $\pi/2$, equals $\rho(\theta)$.

If the first point and final point are coincident, then $c = 0$ and the reference ellipse becomes a circle. In this case, the distribution of the relative course Θ is uniform with density function $f_\Theta(\theta) = 1/2\pi$ where $0 \leq \theta < 2\pi$, the relative course Θ and the length R are independent and the distribution of R is exponential with density function $f_R(r) = (1/\delta)\exp(-r/\delta)$ where $0 < r$.

Consider a process that is defined in the same way except that the length of a track segment is always chosen to be equal to the value determined for R. For this process, a is not the track length, there is not a length constraint, and the final point is not an end point, there is not a termination constraint. The final point is a reference point of the process, literally, a focal point of the process. If $c = 0$, the process is identical to the random tour process described by Washburn in Reference 1. If $c > 0$, it is a generalization of that process.

By inspection of the density function, it can be seen that both the mean and the mode of the relative course Θ are equal to $\pi/2$. For the constrained process, this implies that the most likely next to last and last track segment relative course values are $\pi/2$ and $-\pi/2$. In Appendix 1, it is shown that the expected value of the length R is $a\delta/(a+c)$. For the unconstrained process, this implies that the average track segment length is $a\delta/(a+c)$.

A new process could be generated by redefining the parameters a and c. For example, one could use the definition a = 2c where c is again the distance between the first and final points. In this case the degree of focusing of the process would be constant. However, relative to the process described here, it is conjectured that, for track lengths greater than 2a, the process would concentrate track points about the final point, for track lengths less than 2a, the process would concentrate track points about the first point. In the simulation program, for a track of m segments where m > 2, the maximum length of the first m-2 track segments is computed. And, it is also conjectured that, for a track length less than 2a, the probability that the length of the last track segment would exceed this statistic would be greater than that for the process described here.

III.   The Simulation Program

The simulation program requires three inputs: The distance D between the end points, the track length TL and the maximum expected track segment length δ. The program is for an HP-85 and is written in BASIC for that computer. The program listing is in Appendix 2. After loading the program and initiating RUN, a user is prompted for these inputs and advised of their limits. An example of the prompting with a set of inputs is shown below.

```
DISTANCE (0<D<10)?
10
TRACK LENGTH (0<TL<50)?
50
DELTA (1<δ)?
1
```

Table 1.   The input limits are based on display and memory
           requirements.

As a simulation progresses, the developing track is displayed on the HP-85 CRT. After a simulation is completed, a user is given the option of displaying track data or printing the track and the track data. In the CRT display of a track, the starting point is on the left side and the stopping point is on the right side. In the printed output, a track is rotated 90° so that the starting point is at the top. Figure 4 shows sections of the printed output. In the columned data, I is the track segment number, ϕ is the track segment course for a starting point that is due west of the stopping point, R is the track segment length,

θ is the track segment relative course and X and Y are the coordinates of the last point of the track segment in the first coordinate system. If the starting point is considered to be due west of the stopping point, then the positive x-axis of the first coordinate system is directed east and the positive y-axis is directed north. Recall the origin of the first coordinate system is at the mid-point of the line joining the starting and stopping points.

For the track shown in Figure 4, the termination criterion was satisfied at the simulation of the 49th track segment. In the first line below the columned data, $\underline{\theta}$ is the average track segment relative course where $-90° < \theta \le 270°$, $\underline{R}$ is the average and RM is the maximum track segment length for the first 49 track segments. In the second line, $\Phi(51)$ is the relative course of the 51st and last track segment and R(51) is its length.

If a track simulation reaches 97 segments without termination and the termination criterion is not satisfied for the 98th segment, the simulation is stopped. In this case, the values on the first line below the columned data refer to all of the 98 track segments and "TRUNCATED" is on the second line. For both cases, the average relative course $\underline{\theta}$, the average track length $\underline{R}$ and the maximum track length RM relate to the unconstrained stochastic process.

Figures 5, 6 and 7 show sections of the printed output for three additional simulated tracks. The plots illustrate the effect of the value of δ on the plot scale. Doubling the value of δ doubles the distance between points on the plot. This
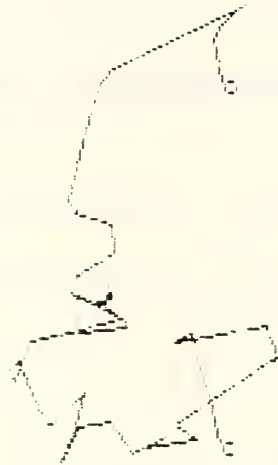
increases the likelihood that the HP-85 CRT will display a complete track. The plots illustrate, to some extent, the increased focusing that occurs as the value of TL approaches the value of D.

A second simulation program to generate track data in a nongraphical but more efficient manner was written. A preliminary analysis of some data generated by the simulation has been made. In Table 2 below, some estimates $\hat{p}$ with associated 95% confidence intervals are given for p, the probability that the last track segment length will exceed the maximum track segment length of the preceding track segments.

D	TL	δ	$\hat{p}$	95% confidence interval
10	50	1	.60	.57 < p < .63
10	50	2	.55	.52 < p < .58
10	50	5	.53	.50 < p < .57
10	20	1	.44	.40 < p < .47
10	20	2	.43	.39 < p < .46
10	20	5	45	.42 < p < .48

Table 2. Estimates $\hat{p}$ for p, the probability that the last track segment length will exceed the maximum track segment length of the preceding track segments. Each estimate is based on a sample size of 900.

DISTANCE=10.00

TRACK LENGTH= 50.00

DELTA= 1.00

I	P	Q	S	X	Y
1	357	1.30	357	-6.17	-.37
2	399	47	301	-6.38	-.04
3	353	07	353	-6.59	.02
48	149	66	217	1.60	1.31
49	170	2.58	149	2.23	-1.35
50	334	52	359	2.01	-.79

Q=099   S=   95   RM= 4.04

P(51)=075   R(51)= 3.10

Figure 4.   A program generated track.   The track data for
segments 4 through 47 have been omitted.

DISTANCE= 5.00

TRACK LENGTH= 50.00

DELTA= 1.00

I	‡	R	∂	X	Y
1	031	2.74	031	-1.09	3.36
2	125	13	092	-.99	2.29
3	062	3.35	039	1.88	3.81
4	210	2.06	129	.85	2.67
32	210	.32	262	60	-5.78
33	339	2.05	050	-.15	-2.67
34	169	17	224	-.11	-4.04

∂=076   R= 1.36   RM= 4.31

‡(35)=033   P(35)= 4.81

Figure 5. A program generated track.

DISTANCE=10.00

TRACK LENGTH= 50.00

DELTA= 2.00

I	φ	R	θ	X	Y
1	236	.39	238	-5.33	-.30
2	113	.91	115	-4.50	-.56
3	139	1.24	132	-3.54	-1.34
25	365	.31	299	2.15	-1.78
26	060	2.01	093	3.90	-.79
27	004	.10	040	3.91	-.69

φ̄=101 R̄= 1.87 RM= 5.52

φ(29)=056 R(29)= 1.39

Figure 6. A program generated track.

```
DISTANCE=10.00
TRACK LENGTH= 20.00
DELTA= 2.00
```


I	Φ	R	θ	X	Y
1	073	3.71	073	-2.40	.73
2	131	.39	115	-2.16	.64
3	200	.43	135	-2.25	.57
17	098	.74	075	1.54	1.48
18	109	1.13	083	3.01	1.11
19	307	.11	377	3.92	1.18

```
R=116   S=    97   Rm= 5.35
S(30)=120   P(30)= 3.39
```

Figure 7. A program generated track.

Appendix 1   Some Mathematical Relationships

In the simulation, values of θ are generated by using the inverse transform method. This method requires the use of the cumulative distribution function $F_\Theta(\theta)$ and it can be found as follows: For $-\pi/2 \le \theta < \pi/2$,

$$F_\Theta(\theta) = \int_{-\pi/2}^{\theta} (1/2\pi)[b/(a-c \sin \theta)] \, d\theta$$

$$= 2[b/(a+c)]\{\tan^{-1}[(a \tan (\theta/2)-c)/b] - \tan^{-1}[-(a+c)/b]\}$$

For $\pi/2 \le \theta < 3\pi/2$, $F_\Theta(\theta) = 1 - F_\Theta(\pi-\theta)$.

The expected value of R for the unmodified process that is given in Section II can be found as follows:

$$E(R) = 2 \int_{-\pi/2}^{\pi/2} E(R|\Theta=\theta) \, f_\Theta(\theta) \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (\delta/\pi) \, b(a-c)[1/(a-c \sin\theta)^2] \, d\theta$$

$$= (2\delta/\pi)[a/(a+c)]\{\tan^{-1}[b/(a+c)] + \tan^{-1}[(a+c)/b]\}$$

$$= a\delta/a+c$$

If the termination criterion is satisfied on the first track segment, the track will consist of only two track segments. With $r_e(\theta) = b^2/(a-c \sin \theta)$, the probability P_2 of this event can be found as follows:

16

$$P_2 = 2 \int_{-\pi/2}^{\pi/2} P(R > r_e(\theta) \mid \Theta = \theta) \ f_\Theta(\theta) \ d\theta$$

$$= 2 \int_{-\pi/2}^{\pi/2} \exp[-r_e(\theta)/\delta(\theta)] \ f_\Theta(\theta) \ d\theta$$

$$= \exp[-(a+c)/\delta]$$

Appendix 2. The HP-85 Simulation Program

```
10 COM F(100),R(100),T(100),X(1
   00),Y(100)
20 DISP "DISTANCE";" (0";CHR$(1
   88);"D";CHR$(188);"10)";
30 INPUT D@ C1=D/2
40 IF D>10 OR D<0 THEN 20
50 DISP "TRACK LENGTH";" (0";CH
   R$(60);"TL";CHR$(188);"50)";
60 INPUT L
70 IF L>50 OR L<=0 THEN 50 ELSE
   A1=L/2
80 DISP "DELTA";" (1";CHR$(188)
   ;CHR$(18);")";
90 INPUT D1
100 IF D1<=0 THEN 80
110 RANDOMIZE
120 X(0)=-C1 @ Y(0)=0 @ F(0)=0 @
    R(0)=0 @ T(0)=0
130 B1=0 @ B2=0 @ B3=0 @ R1=0 @
    A=A1 @ C=C1 @ S=0 @ V=0 @ W=
    0
140 GCLEAR @ E=10*SQR(D1)
150 SCALE -E,E,-(.75*E),.75*E
160 MOVE C1,0 @ LABEL "o"
170 MOVE -C1,0 @ LABEL "o"
180 FOR I=1 TO 98
190 U=RND @ G1=0
200 IF U<=.5 THEN 220
210 U=U-.5 @ G1=1
220 N1=SQR((A1+C1)/(A1-C1)) @ N2
    =SQR(1-C1*C1/(A1*A1))
230 T=2*ATN(N2*TAN(PI*U-ATN(N1))
    )+C1/A1
240 IF G1=1 THEN T=PI-T
250 P=(A1-C1)/(A1-C1*SIN(T))
260 D2=D1*P
270 R=-(D2*LOG(1-RND))
280 F=T+S
290 GOSUB 800
300 IF A2>C2 THEN 340
310 R=(A*A-C*C)/(A-C*SIN(T))
320 GOSUB 800
330 B2=1
340 DRAW X(I),Y(I)
350 A=A2 @ C=C2 @ S=-ATN2(Y,X) @
    R(I)=R
360 H=T @ GOSUB 770
370 T(I)=H
380 H=F @ GOSUB 770
390 F(I)=H
400 IF B2=1 THEN 450
410 R1=MAX(R,R1)
420 V=V+(T-W)/I @ W=W+(R-W)/I
430 NEXT I
440 DISP @ DISP "TRUNCATED" @ K=
    I @ B3=1 @ GOSUB 850 @ GOTO
    480
450 DRAW C1,0
460 K=I+1 @ R(K)=2*C @ T(K)=90 @
    H=S+PI/2 @ GOSUB 770 @ F(K)
    =H
```

18

```
470 GOSUB 850
480 DISP @ DISP "DISP DATA (Y/N)
    ",
490 INPUT A$
500 IF A$="Y" THEN 520
510 GOTO 680
520 DISP USING 530 ; CHR$(15),CH
    R$(16)
530 IMAGE ,3/,X,"I",3X,A,4X,"R",
    5X,A,5X,"X",6X,"Y"
540 FOR I=1 TO K-1
550 DISP USING 560 ; I,F(I),R(I)
    ,T(I),X(I),Y(I)
560 IMAGE ,/,2D,2X,3Z,X,2D 2D,2X
    ,3Z,X,3D 2D,X,3D.2D
570 NEXT I
580 DISP USING 590 ; CHR$(144),V
    ,CHR$(210),W,R1
590 IMAGE ,/,A,"=",3Z,2X,A,"=",
    2D 2D,2X,"RM=",2D 2D
600 IF B3=1 THEN 640
610 DISP USING 620 ; CHR$(15),K,
    F(K),K,R(K)
620 IMAGE ,/,A,"(",2D,")=" 3Z,2X
    ,"R(",2D,")=",2D 2D,9/,
630 GOTO 660
640 DISP USING 650 ; "TRUNCATED"
650 IMAGE ,/,9A,9/,
660 IF B1=0 THEN 680
670 CRT IS 1 @ GOTO 710
680 DISP @ DISP "PRINT PLOT & DA
    TA (Y/N)";
690 INPUT B$
700 IF B$="Y" THEN 720
710 ALPHA @ END
720 PRINT USING 730 ; D,L,D1
730 IMAGE ,3/,"DISTANCE=",2D.DD,
    ,/,"TRACK LENGTH=",DDD.DD,,/,
    ,"DELTA=",DD.DD,3/
740 CRT IS 2 @ B1=1
750 GRAPH @ COPY
760 GOTO 520
770 H=RTD(H) @ H=RMD(H,360)
780 IF H<0 THEN H=H+360
790 RETURN
800 X(I)=X(I-1)+R*SIN(F)
810 Y(I)=Y(I-1)+R*COS(F)
820 X=C1-X(I) @ Y=-Y(I)
830 C2=SQR(X*X+Y*Y)/2 @ A2=A-R/2
840 RETURN
850 H=V @ H=RTD(H) @ V=H
860 RETURN
```

REFERENCES

1. Washburn, Alan R., "Probability Density of a Moving Particle", Operations Research, 17 (1969), 861-871.

INITIAL DISTRIBUTION LIST

Dean of Research, Code 012 Naval Postgraduate School Monterey, CA 93940	1
Defense Technical Information Center Cameron Station Alexandria, VA 22314	2
Library, Code 0212 Naval Postgraduate School Monterey, CA 93940	2
Director Strategic Systems Projects Office Attn: Code SP-2024, Code SP-20231 1931 Jefferson Davis Highway Arlington, VA 20376	5
Applied Physics Laboratory Attn: R. N. McDonough Johns Hopkins Road Laurel, MD 20707	1
EPL Analysis 3460 Olney-Laytonsville Road Olney, MD 20832	1
ORI, Inc. Attn: P. L. Hallowell 1400 Spring St. Silver Spring, MD 20910	1
Pacific Sierra Research Corp. Attn: E. L. Holmboe 1401 Wilson Blvd. Arlington, VA 22209	1
R and D Associates Attn: A. G. Cicolani 1401 Wilson Blvd., Suite 500 Arlington, VA 22209	1
D. H. Wagner, Associates Attn: B. Belkin Station Square One Paoli, PA 19301	1
D. C. Bossard Inc. 1107 Montgomery Ave. Rosemont, PA 19010	1

Corwin Analysis   1
2609 Crum Creek Dr.
Berwyn, PA   19312

ORINCON, Inc.   1
Attn: D. Alspach
3366 North Torrey Pines Crt.
Suite 320
La Jolla, CA   92037

R. N. Forrest, Code 71   20
Naval Postgraduate School
Monterey, CA   93940

U206426